Spring 2017 MATH5012

Real Analysis II

Solution to Exercise 4

(1) Let ω_n be the volume of the unit ball in $\mathbb{R}^{n+1}$, so $\omega_1 = 2, \omega_2 = \pi, \omega_3 = 4/3\pi$, etc. Show that

$$\omega_n = 2\omega_{n-1} \int_0^1 (1-x^2)^{(n-1)/2} dx,$$

and deduce the formula

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \ .$$

Look up the definition of the Gamma function yourself. This is supposed a problem on Fubini's theorem in advanced calculus.

Solution. Note that the Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, x \in (0, \infty).$$

Let

$$I = \int_{\mathbb{R}^n} e^{-|x|^2} dx_1 \cdots dx_n.$$

We calculate I in two ways. First, by Fubini's theorem,

$$I = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-x_1^2} \cdots e^{-x_n^2} dx_1 \cdots dx_n = \Big(\int_{-\infty}^\infty e^{-t^2} dt \Big)^n = \big(\sqrt{\pi}\big)^n.$$

On the other hand, expressing I in the polar coordinates,

$$\begin{aligned} I &= \int_{S_1}\int_0^\infty e^{-r^2}r^{n-1}drd\theta \\ &= \frac{|S_1|}{2}\int_0^\infty e^{-t}t^{n/2-1}dt \\ &= \frac{n|B_1|}{2}\times\Gamma\Big(\frac{n}{2}\Big) \\ &= |B_1|\Gamma\Big(1+\frac{n}{2}\Big), \end{aligned}$$

and the results follows.

(2) Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt \quad (x>0)$$

to prove that

$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Solution. Put $f(x,y) = \sin x e^{-xt}$. Observe that f is $\mathcal{L}^2$-measurable and $|f(x,y)| \le e^{-xt}$ whose iterated integral over $[0,\infty)^2$ is finite. $f \in L^2$. By Fubini's theorem,

$$\begin{aligned} \int_0^A \frac{\sin x}{x}dx &= \int_0^A\int_0^\infty \sin x e^{-xt}dtdx \\ &= \int_0^\infty\int_0^A \sin x e^{-xt}dxdt \\ &= \int_0^\infty \frac{-t\sin Ae^{-At}}{t^2+1} - \frac{\cos Ae^{-At}}{t^2+1} + \frac{1}{t^2+1}dt \end{aligned}$$

But if $A \ge 1$,

$$\left|\frac{-t\sin Ae^{-At}}{t^2+1} - \frac{\cos Ae^{-At}}{t^2+1}\right| \le 2e^{-At} \le 2e^{-t};$$

by the dominated convergence theorem,

$$\lim_{A\to\infty}\int_0^\infty \frac{-t\sin Ae^{-At}}{t^2+1} - \frac{\cos Ae^{-At}}{t^2+1} + \frac{1}{t^2+1}dt = \int_0^\infty \frac{1}{t^2+1}dt = \frac{\pi}{2}.$$

This integral can also be evaluated using residues in complex analysis.

(3) Complete the following proof of Hardy's inequality (chapter 3, Exercise 14 in [R1]): Suppose $f \geq 0$ on $(0,\infty)$, $f \in L^p$, $1 < p < \infty$, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt.$$

Write $xF(x) = \int_0^x f(t)t^\alpha t^{-\alpha}\,dt$, where $0 < \alpha < 1/q$, use Hölder's inequality to get an upper bound for $F(x)^p$, and integrate to obtain

$$\int_0^\infty F^p(x)\,dx \leq (1-\alpha q)^{1-p}(\alpha p)^{-1}\int_0^\infty f^p(t)\,dt.$$

Show that the best choice of α yields

$$\int_0^\infty F^p(x)\,dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)\,dt.$$

Solution. If $0 < \alpha < \frac{1}{q}$.

$$\begin{aligned} xF(x) &= \int_0^x f(t)t^\alpha t^{-\alpha}dt \\ &\leq \left(\int_0^x f^p(t)t^{\alpha p}dt\right)^{\frac{1}{p}} \left(\int_0^x t^{-\alpha q}dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^x f^p(t)t^{\alpha p}dt\right)^{\frac{1}{p}} \left(\frac{x^{-\alpha q+1}}{-\alpha q+1}\right)^{\frac{1}{q}}. \end{aligned}$$

Thus we have

$$F^p(x) \leq (1-\alpha q)^{1-p}x^{-1-\alpha p}\int_0^x f^p(t)t^{\alpha p}dt.$$

Integrating gives

$$\begin{aligned}\int_0^\infty F^p(x)dx &\le (1-\alpha q)^{1-p}\int_0^\infty x^{-1-\alpha p}\int_0^x f^p(t)t^{\alpha p}dtdx \\ &= (1-\alpha q)^{1-p}\int_0^\infty\int_0^\infty \chi_{(0,x)}(t)x^{-1-\alpha p}f^p(t)t^{\alpha p}dxdt \text{ (applied Fubini's)} \\ &= (1-\alpha q)^{1-p}\int_0^\infty f^p(t)t^{\alpha p}\int_t^\infty x^{-1-\alpha p}dxdt \\ &= (1-\alpha q)^{1-p}(\alpha p)^{-1}\int_0^\infty f^p(t)dt\end{aligned}$$

To minimize the constant, we would like to choose a smallest α. A simple computation shows that

$$\frac{d}{d\alpha}(1-\alpha q)^{1-p}(\alpha p)^{-1} = \frac{-p(1-\alpha q)^{p-1}+\alpha p^2(1-\alpha q)^{p-2}}{\alpha^2p^2(1-\alpha q)^{2p-2}} = \frac{\alpha(p+q)-1}{\alpha^2 p(1-\alpha q)^p}$$

and

$$\frac{d}{d\alpha}(1-\alpha q)^{1-p}(\alpha p)^{-1} \text{ iff } \alpha = \frac{1}{p+q}.$$

Obviously, if $0<\alpha<\frac{1}{p+q}$,

$$\frac{d}{d\alpha}(1-\alpha q)^{1-p}(\alpha p)^{-1} < 0$$

and if $\frac{1}{p+q}<\alpha<\frac{1}{q}$,

$$\frac{d}{d\alpha}(1-\alpha q)^{1-p}(\alpha p)^{-1} > 0.$$

It follows that $(1-\alpha q)^{1-p}(\alpha p)^{-1}$ attains minimum at $\alpha=\frac{1}{p+q}$ whose value is $q^p = \left(\frac{p}{p-1}\right)^p$ and consequently,

$$\int_0^\infty F^p(x)dx \le \left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(t)dt.$$

(4) Prove the following analogue of Minkowski's inequality, for $f \geq 0$:

$$\left\{\int \left[\int f(x,y)\, d\lambda(y)\right]^p d\mu(x)\right\}^{\frac{1}{p}} \leq \int \left[\int f^p(x,y)\, d\mu(x)\right]^{\frac{1}{p}} d\lambda(y).$$

Supply the required hypotheses.

Solution. Suppose that $p > 1$, μ, λ are σ-finite. The case $p = 1$ is elementary. We assume, since otherwise trivial, that

$$M = \int \left(\int f^p(x,y) d\mu(x)\right)^{\frac{1}{p}} d\lambda(y) < \infty.$$

Put

$$h(x) = \int f(x,y) d\lambda(y)$$

and define the linear functional Λ on $L^q(\mu)$ by

$$\Lambda g = \int hg d\mu \ , \quad \forall g \in L^q(\mu).$$

To show that Λ is bounded, we note

$$\begin{aligned}
|\Lambda(g)| &= \int\int f(x,y)|g(x)| d\lambda(y) d\mu(x) \\
&= \int\int f(x,y)|g(x)| d\mu(x) d\lambda(y) \quad (\text{ Fubini }) \\
&\leq \int \left(\int f^p(x,y) d\mu(x)\right)^{\frac{1}{p}} \|g\|_{L^q(\mu)} d\lambda(y) \quad (\text{ Hölder }) \\
&= \|g\|_q \int \left(\int f^p(x,y) d\mu(x)\right)^{\frac{1}{p}} d\lambda(y) \\
&= M\|g\|_q < \infty.
\end{aligned}$$

By duality, we have

$$\|h\|_p = \|\Lambda\|_{op} = \sup_{\{g, \|g\|_q = 1\}} |\Lambda g| \leq M \ ,$$

done.

Many problems are taken from chapter 8, [R1].